Appendix A. Supplementary Material to "Fighting Collusion by Permitting Price Discrimination" by M. Helfrich and F. Herweg

This appendix contains detailed proofs to the propositions and lemmas presented in the paper "Fighting Collusion by Permitting Price Discrimination".

Proof of Lemma 1. First, we consider the static Nash equilibrium. The profit function of firm B is (symmetric for firm A)

$$\pi_B = (p_B - c)\left\{\alpha\left[\bar{\theta} - \frac{p_B - p_A}{2\rho_L}\right] + (1-\alpha)\left[\bar{\theta} - \frac{p_B - p_A}{2\rho_H}\right]\right\}\frac{1}{2\bar{\theta}}. \tag{A.1}$$

The first-order condition of profit maximization is

$$\alpha\left[\bar{\theta} - \frac{p_B - p_A}{2\rho_L}\right] + (1-\alpha)\left[\bar{\theta} - \frac{p_B - p_A}{2\rho_H}\right] - (p_B - c)\left[\frac{\alpha}{2\rho_L} + \frac{1-\alpha}{2\rho_H}\right] = 0. \tag{A.2}$$

In the symmetric equilibrium, each firm sets the price $p_U^N = c + 2R\bar{\theta}$. It can readily be established that no asymmetric equilibrium exists. Inserting the equilibrium price in the profit function yields the Nash equilibrium profit $\pi_U^N = R\bar{\theta}$.

The price p_U^N indeed constitutes an equilibrium of the static game only if all consumers purchase either from firm A or firm B. Type $\theta = 0$ prefers to buy one unit (either from A or from B) instead of not buying a good if $v - p_U^N > 0$. This condition is equivalent to

$$v - c > \frac{2\rho_L\rho_H\bar{\theta}}{(1-\alpha)\rho_L + \alpha\rho_H}, \tag{A.3}$$

which holds by assumption (by (9)).

If the firms collude and it is optimal to serve all consumers, then the optimal price is $p_U^C = v$ leading to a firm profit of $\pi_U^C = (v-c)/2$. It might, however, be optimal not to serve all consumers; i.e., to charge a price $p > v$ so that some types with θ close to zero purchase from neither firm. For prices $p > v$ each firm is a (local) monopolist and thus a consumer of type $\theta = (p_B - v)/\rho_k$ is indifferent between purchasing from B and not purchasing the good. The profit of firm B is given by

$$\pi_B = (p_B - c)\left\{\alpha\left[\bar{\theta} - \frac{p_B - v}{\rho_L}\right] + (1-\alpha)\left[\bar{\theta} - \frac{p_B - v}{\rho_H}\right]\right\}\frac{1}{2\bar{\theta}}. \tag{A.4}$$

From the first-order condition the optimal price is readily obtained,

$$p^* = \frac{1}{2}\left(v + c + \frac{\rho_L \rho_H}{(1-\alpha)\rho_L + \alpha\rho_H}\bar{\theta}\right). \tag{A.5}$$

This price – and thus not serving all consumers – is optimal only if it is higher than the willingness to pay v. Note that $p^* > v$ is equivalent to

$$v - c < \frac{\rho_L \rho_H \bar{\theta}}{(1-\alpha)\rho_L + \alpha\rho_H}, \tag{A.6}$$

which is never satisfied under the imposed assumption (by (9)).

Next, we consider the optimal deviation from the collusive agreement. From the first-order condition (A.2) we obtain that the best response to $p_A = v$ is

$$p_U^D = \frac{1}{2}(v + c) + \frac{\rho_L \rho_H \bar{\theta}}{(1-\alpha)\rho_L + \alpha\rho_H}. \tag{A.7}$$

This price is derived under the presumption that the marginal consumers are interior. This is indeed the case if and only if for all $k \in \{L, H\}$ it holds that $-\bar{\theta} < \hat{\theta}_k$. This condition can be written as

$$2\rho_k \bar{\theta} > \frac{1}{2}(v - c) - \frac{\rho_L \rho_H}{(1-\alpha)\rho_L + \alpha\rho_H}\bar{\theta} \quad \forall k \in \{L, H\}. \tag{A.8}$$

The above condition is hardest to satisfy for $k = L$. Setting $\rho_k = \rho_L$ in (A.8) and rearranging yields

$$v - c < \frac{(4-4\alpha)\rho_L + (2+4\alpha)\rho_H}{(1-\alpha)\rho_L + \alpha\rho_H}\rho_L \bar{\theta}. \tag{A.9}$$

Finally, we derive the critical discount factor. The critical discount factor is defined by $\bar{\delta}_U = (\pi_U^D - \pi_U^C)/(\pi_U^D - \pi_U^N)$. Hence,

$$\begin{aligned}\bar{\delta}_U &= \frac{\frac{1}{4R\bar{\theta}}\left[\frac{1}{2}(v-c) + R\bar{\theta}\right]^2 - \frac{1}{2}(v-c)}{\frac{1}{4R\bar{\theta}}\left[\frac{1}{2}(v-c) + R\bar{\theta}\right]^2 - R\bar{\theta}} \\ &= \frac{\left[\frac{1}{2}(v-c) - R\bar{\theta}\right]^2}{\left[\frac{1}{2}(v-c) + R\bar{\theta}\right]^2 - 4R^2\bar{\theta}^2},\end{aligned} \tag{A.10}$$

which concludes the proof. □

Proof of Lemma 2. The overall profit of firm $i = A, B$ is given by the sum of profits from consumer groups L and H; i.e. $\pi_{DS,i} = \sum_{k\in\{L,H\}} \pi_i^k$. Profits can be maximized independently for each consumer group. Note that the maximization problem for one consumer group k is equivalent to the maximization problem with uniform pricing and $\rho_L = \rho_H = \rho_k$.

First, we consider the static Nash equilibrium. Note that with $\rho_L = \rho_H = \rho_K$, the expression defined as R in (6) simplifies to ρ_k. Thus, in the static Nash equilibrium, each firm charges the price $p_{DS}^{N,k} = c + 2\rho_k\bar{\theta}$ and makes a profit of $\pi_{DS}^k = \omega_k\rho_k\bar{\theta}$ from selling to consumer group k. Overall profit is given by the sum of profits from both groups, i.e. $\pi_{DS}^N = \alpha\rho_L\bar{\theta} + (1-\alpha)\rho_H\bar{\theta}$.

The price $p_{DS}^{N,k} = c + 2\rho_k\bar{\theta}$ constitutes an equilibrium of the static game only if all consumers purchase either from firm A or from firm B. Type $\theta = 0$ and $\rho = \rho_H$, who has the lowest utility, still prefers to buy one unit of the good if $v - p_{DS}^{N,H} > 0$. This is equivalent to $v - c > 2\rho_H\bar{\theta}$, which holds by (15).

If the firms collude and if it is optimal to serve all consumers, the profit maximizing price is $p_{DS}^C = v$ for both consumer groups. Both firms earn an overall profit of $\pi_{DS}^C = (v-c)/2$. It could, however, be optimal not to serve all consumers in a group, i.e. to set a price higher than v so that some consumers do not purchase the good. Suppose both firms agree on prices $p_i^k > v$. Firm B's profit from type ρ_k consumers now is given by $\pi_B^k = \omega_k(p_B^k - c)\left[\bar{\theta} - (p_B^k - v)/\rho_k\right]/(2\bar{\theta})$. The optimal price is $p_B^{k*} = (v + c + \rho_k\bar{\theta})/2$. It is optimal not to serve all consumers only if $p_B^{k*} > v$, which is equivalent to $\rho_k\bar{\theta} > v - c$. Under assumption (15), this condition is not satisfied.

Consider next firm B unilaterally deviating from the collusive agreement. For each consumer group k, firm B maximizes the profit function π_B^k given that A charges the cartel price $p_{DS}^C = v$. The best response to the rival firm charging the cartel price is $p_{DS}^{D,k} = (v+c)/2 + \rho_k\bar{\theta}$. With $p_{DS}^{D,k}$, we have interior marginal consumers only if for all $k \in L, H$ it holds that $\hat{\theta}_k > -\bar{\theta}$. After inserting prices, this condition can be written as $v - c < 6\rho_k\bar{\theta}$, which is harder to satisfy for $k = L$. Thus, the deviation price we computed is indeed optimal if and only if $v - c < 6\rho_L\bar{\theta}$, which is satisfied under assumption (15).

Now, we can compute the critical discount factor $\bar{\delta}_{DS} = (\pi_{DS}^D - \pi_{DS}^C)/(\pi_{DS}^D -$

π_{DS}^{N}):

$$\bar{\delta}_{DS} = \frac{\sum_{k\in\{L,H\}} \frac{\omega_k}{4\rho_k\bar{\theta}} \left[\frac{1}{2}(v-c) + \rho_k\bar{\theta}\right]^2 - \frac{1}{2}(v-c)}{\sum_{k\in\{L,H\}} \frac{\omega_k}{4\rho_k\bar{\theta}} \left[\frac{1}{2}(v-c) + \rho_k\bar{\theta}\right]^2 - \omega_k\rho_k\bar{\theta}}$$
$$= \frac{\sum_{k\in\{L,H\}} \frac{\omega_k}{\rho_k} \left[\frac{1}{2}(v-c) - \rho_k\bar{\theta}\right]^2}{\sum_{k\in\{L,H\}} \frac{\omega_k}{\rho_k} \left[\frac{1}{4}(v-c)^2 + (v-c)\rho_k\bar{\theta} - 3\rho_k^2\bar{\theta}^2\right]} \tag{A.11}$$

□

Proof of Proposition 1. The critical discount factors, $\bar{\delta}_U$ and $\bar{\delta}_{DS}$, are given by (10) and (16), respectively, only if the constraints (9) and (15) are jointly satisfied. It is easy to verify that (15) sets both the more restrictive lower and upper bound on $v - c$; i.e. whenever (15) is satisfied, (9) is fulfilled too. To prove that $\bar{\delta}_U < \bar{\delta}_{DS}$, we show that the fraction defining $\bar{\delta}_{DS}$ has a larger numerator (N_j) and smaller denominator (D_j) than the fraction defining $\bar{\delta}_U$, with $j \in \{U, DS\}$.

We start with the numerator. As cartel profits are the same in both cases, $N_{DS} - N_U > 0$ is equivalent to $\pi_{DS}^D - \pi_U^D > 0$.

$$\pi_{DS}^D - \pi_U^D > 0$$
$$\iff \alpha \left[\frac{1}{\rho_L}\left(\frac{1}{2}(v-c) + \rho_L\bar{\theta}\right)^2\right] + (1-\alpha)\left[\frac{1}{\rho_H}\left(\frac{1}{2}(v-c) + \rho_H\bar{\theta}\right)^2\right]$$
$$- \frac{1}{R}\left[\frac{1}{2}(v-c) + R\bar{\theta}\right]^2 > 0. \tag{A.12}$$

Condition (A.12) can be simplified to $(\rho_H - \rho_L)^2 > 0$, which is always satisfied.

Next, consider the denominator. $D_U - D_{DS} > 0$ is equivalent to

$$\frac{1}{4\bar{\theta}R}\left[\frac{1}{2}(v-c) + R\bar{\theta}\right]^2 - R\bar{\theta} - \frac{\alpha}{4\bar{\theta}\rho_L}\left[\frac{1}{2}(v-c) + \rho_L\bar{\theta}\right]^2$$
$$- \frac{1-\alpha}{4\bar{\theta}\rho_H}\left[\frac{1}{2}(v-c) + \rho_H\bar{\theta}\right]^2 + \alpha\rho_L\bar{\theta} + (1-\alpha)\rho_H\bar{\theta} > 0. \tag{A.13}$$

Condition (A.13) can be simplified to $(\rho_H - \rho_L)^2 > 0$, which is always satisfied.

□

Proof of Lemma 3. First, we consider the static Nash equilibrium. Firms maximize their profit in the (–)-and in the (+)-market independently. The profit functions of firms A and B in the (–)-market are

$$\pi^-_{DA,A} = (p^-_A - c)\left(\bar{\theta} + \frac{p^-_B - p^-_A}{2\rho}\right)\frac{1}{2\bar{\theta}} \tag{A.14}$$

$$\pi^-_{DA,B} = (p^-_B - c)\left(-\frac{p^-_B - p^-_A}{2\rho}\right)\frac{1}{2\bar{\theta}}. \tag{A.15}$$

From the two first-order conditions, we get the equilibrium prices $p^{N-}_{DA,A} = c + (4/3)\rho\bar{\theta}$, $p^{N-}_{DA,B} = c + (2/3)\rho\bar{\theta}$ and profits $\pi^{N-}_{DA,A} = (4/9)\rho\bar{\theta}$, $\pi^{N-}_{DA,B} = (1/9)\rho\bar{\theta}$. The (+)-market is symmetric to the (−)-market, only firms' roles are reversed. Thus, the equilibrium prices and profits are $p^{N+}_{DA,A} = c+(2/3)\rho\bar{\theta}$, $p^{N+}_{DA,B} = c + (4/3)\rho\bar{\theta}$, $\pi^{N+}_{DA,A} = (1/9)\rho\bar{\theta}$, and $\pi^{N+}_{DA,B} = (4/9)\rho\bar{\theta}$.

Firm i's overall profit is given by

$$\pi^N_{DA,i} = \pi^{N-}_{DA,i} + \pi^{N+}_{DA,i} = \frac{5}{9}\rho\bar{\theta}. \tag{A.16}$$

The prices $p^{N,z}_{DA,i}$ constitute Nash equilibria only if all consumers purchase the good. Note that there are two marginal consumers, who are indifferent between buying from A or B: $\hat{\theta}^- = -\bar{\theta}/3$ and $\hat{\theta}^+ = \bar{\theta}/3$. The marginal consumers obtain the lowest utility from purchasing a good. Thus, if the utility from purchasing either good of the marginal consumer is positive, then all consumer types obtain a strictly positive utility from buying. The utility of the marginal consumers is positive if $v - c > \rho\bar{\theta}$, which is fulfilled under assumption (21).

If the firms form a cartel, the profit maximizing price is $p^C_{DA} = v$ for both markets. At this price, firm A serves the (−)-market and B the (+)-market. Each firm earns an overall profit of $\pi^C_{DA} = (v - c)/2$. To check that setting p^C_{DA} and serving all consumers is indeed optimal, suppose, for example, that firm B sets a price $p^+_B > v$ in the (+)-market (analogous for A and the (−)-market). The profit is given by $\pi^+_B = (p^+_B - c)\left[\bar{\theta} - (p^+_B - v)/\rho\right]/(2\bar{\theta})$. The optimal price is $p^*_B = (v + c + \rho\bar{\theta})/2$. It is optimal not to serve all consumers only if $p^*_B > v$, which is equivalent to $v - c < \rho\bar{\theta}$. However, this condition is not satisfied under assumption (21).

Consider next one firm deviating from the collusive agreement. Due to price discrimination, a firm can gain demand and thereby increase its profit by setting a lower price in the market served by the competitor while still charging the cartel price p_{DA}^C and making the cartel profit $\pi_{DA}^C = (v-c)/2$ in its strong market. The profit from deviation is

$$\frac{1}{2}(v-c) + \left(p^D - c\right)\left(\frac{v-p^D}{2\rho}\right)\frac{1}{2\bar{\theta}}. \tag{A.17}$$

From the first-order condition, we get the optimal deviation price $p_{DA}^D = (v+c)/2$. Inserting this price in the profit function yields the deviating firm's overall profit:

$$\pi_{DA}^D = \frac{1}{2}\left[(v-c) + \frac{1}{8\rho\bar{\theta}}(v-c)^2\right]. \tag{A.18}$$

With the deviation price, we have interior marginal consumers only if $\hat{\theta} > -\bar{\theta}$. After inserting prices, this condition simplifies to $v - c < 4\rho\bar{\theta}$, which is satisfied by assumption (21).

Finally, we can compute the critical discount factor $\bar{\delta}_{DA} = (\pi_{DA}^D - \pi_{DA}^C)/(\pi_{DA}^D - \pi_{DA}^N)$:

$$\begin{aligned}\bar{\delta}_{DA} =& \frac{\frac{1}{2}(v-c) + \frac{1}{16\rho\bar{\theta}}(v-c)^2 - \frac{1}{2}(v-c)}{\frac{1}{2}(v-c) + \frac{1}{16\rho\bar{\theta}}(v-c)^2 - \frac{5}{9}\rho\bar{\theta}} \\ =& \frac{(v-c)^2}{8\rho\bar{\theta}(v-c) + (v-c)^2 - \frac{80}{9}\rho^2\bar{\theta}^2}.\end{aligned} \tag{A.19}$$

□

Proof of Proposition 2. To compare the critical discount factors, the assumptions on $\bar{\delta}_U$ from (9) as well as those on $\bar{\delta}_{DA}$ from (21) have to be fulfilled. For $\rho_L = \rho_H = \rho$, (9) simplifies to $2\rho\bar{\theta} < v - c < 6\rho\bar{\theta}$, which implies that (9) sets the more restrictive lower bound, and (21) the more restrictive upper bound on $v - c$. Thus, $v - c$ is restricted to $2\rho\bar{\theta} < v - c < 4\rho\bar{\theta}$.

We want to show that $\bar{\delta}_U < \bar{\delta}_{DA}$, which is equivalent to

$$\frac{\left[\frac{1}{2}(v-c) - \rho\bar{\theta}\right]^2}{\frac{1}{4}(v-c)^2 + \rho\bar{\theta}(v-c) - 3\rho^2\bar{\theta}^2} < \frac{(v-c)^2}{8\rho\bar{\theta}(v-c) + (v-c)^2 - \frac{80}{9}\rho^2\bar{\theta}^2}, \tag{A.20}$$

where we simplified $\bar{\delta}_U$ setting $\rho_L = \rho_H = \rho$. To simplify our expressions, we define $v - c =: x$ and $\rho\bar{\theta} =: y$. Substituting $v - c$ and $\rho\bar{\theta}$ with x and y and rearranging (A.20) yields

$$x^2 - \frac{19}{7}xy + \frac{10}{7}y^2 > 0. \tag{A.21}$$

Note that the left-hand side of (A.21) is a function of x with a U-shaped graph. This function is equal to zero for $x_1 = (5/7)\rho\bar{\theta}$ and $x_2 = 2\rho\bar{\theta}$; it takes on negative values in the interval (x_1, x_2) and is positive otherwise. Hence, under the considered restrictions on $v - c$ it always holds that $\bar{\delta}_U < \bar{\delta}_{DA}$.[5] $\square$

[5] If the restrictions on $v - c$ are relaxed – in particular the lower bound – , then this does not imply a reversed ordering of the critical discount factors because then $\bar{\delta}_U$ is no longer given by (10).